LINEAR AND NONLINEAR MECHANISMS OF INFORMATION PROPAGATION

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Abstract. The mechanisms of information transmission are investigated in a lattice of coupled continuous maps, by analyzing the propagation of both finite and infinitesimal disturbances. Two distinct regimes are detected: in the former case, both classes of perturbations spread with the same velocity; in the latter case, finite perturbations propagate faster than infinitesimal ones. The transition between the two phases is also investigated by determining the scaling behaviour of the order parameter.

PACS No: 05.45+b

Recently, it has been discovered that coupled-map lattices (CML) can exhibit irregular behaviour even if the maximum Lyapunov exponent is negative [1, 2]. This phenomenon has been called "stable chaos" [2] in order to stress the apparent incongruity. As a matter of fact, a periodic configuration is always approached in a finite time, on a lattice of length L. However, the average transient time $T(L)$ diverges exponentially with L. In the thermodynamic limit $L \to \infty$, the statistical behaviour exhibited for times shorter than $T(L)$ appears to be stationary. Therefore, the usual chaotic indicators (as, e.g., Lyapunov exponents) can be meaningfully applied to the characterization of the transient evolution as well. In analogy with standard transient chaos, the value of the Lyapunov exponents depends on the order the limits $L \to \infty$ and $t \to \infty$ are taken.

The reference model considered in this Letter is a coupled-map lattice based on the usual diffusive-coupling scheme [3],

$$x^i_{t+1} = f(\bar{x}^i_t) \tag{1a}$$

$$\bar{x}^i_t = \frac{\varepsilon}{2} x^{i-1}_t + (1-\varepsilon) x^i_t + \frac{\varepsilon}{2} x^{i+1}_t \tag{1b}$$

where t is the discrete time, i labels the lattice sites, ε measures the coupling strength and periodic boundary conditions are assumed ($x^0_t = x^L_t$, $x^{L+1}_t = x^1_t$).

In both [1] and [2], the map f responsible for stable chaos is of the type

$$f(x) = \begin{cases} p_1 x + q_1 & 0 \le x \le x_c \\ 1 - (1-q_2)(x - x_c)/\eta & x_c < x \le x_c + \eta \\ q_2 + p_2(x - x_c - \eta) & x_c + \eta < x \le 1 \end{cases} , \tag{2}$$

($x_c = (1-q_1)/p_1$) for $\eta = 0$, i.e. it is discontinuous and piecewise linear. The single map considered in Ref. [1] (f_1), corresponding to the parameter values $p_1 = p_2 = 0.91$, $q_1 = .1$, and $q_2 = 0$, is characterized by a globally stable period-25 orbit. The parameters of the map in Ref. [2] (f_2) are $p_1 = 2.7$, $p_2 = 0.1$, $q_1 = 0$ and $q_2 = 0.07$. In such a case, a globally stable period-3 orbit exists.

One can easily conjecture that the sudden amplification of the distance between points lying across the discontinuity (a phenomenon which escapes the linear stability analysis) is at the origin of the random evolution. In fact, Ershov

and Potapov [4] have shown that nonchaotic turbulence persists in the CML if the map f_1 is made continuous by introducing a finite $\eta < 4 \cdot 10^{-4}$. However, they have also shown that the continuization of the map destabilizes the period-25 orbit, leading to a standard chaotic attractor in the single map for $\eta > 5 \cdot 10^{-3}$ Therefore, they have concluded that the observed irregular behaviour is a reminiscence of the usual transient chaos, characterized by a positive Lyapunov exponent.

We have found that, in analogy to what happens for the model of Ref. [1], the dynamics of the CML with a continuous f_2 exhibits a negative maximum Lyapunov exponent if $\eta < \eta_1 = 4 \cdot 10^{-4}$. However, in our case, the critical value of η corresponding to the destabilization of the period-3 orbit is much higher ($\eta_2 = p_1^2 q_2 - 1/p_1 \simeq 0.14$). Therefore, the phenomenon of stable chaos has a less obvious origin than conjectured in Ref. [4]. Nevertheless, one must conclude that in both models, the linear stability analysis is not able to capture the essence of the observed irregular behaviour.

We conjecture that two different mechanisms of information flow on the lattice must be invoked: one based on the usual (local) linear instability, the other based on "nonlinear" propagation phenomena, analogous to those present in cellular automata [5]. The aim of the present Letter is to perform a comparative study of the two mechanisms.

The CML has been analyzed for $\varepsilon = 2/3$. We investigate the evolution of an initially localized disturbance $w_{t=0}^i = \delta_{i,0}$ in the linear approximation. Its time evolution is of the type

$$w_t^i \simeq \mathrm{e}^{\Lambda(i/t)t} \quad , \tag{3}$$

where the comoving Lyapunov exponent $\Lambda(v)$ [6] is the growth rate of the disturbance in a reference frame moving with velocity $v = i/t$. Within the light-cone defined by

$$\Lambda(v_\ell) = 0 \quad ,$$

the perturbation is exponentially amplified. Accordingly, v_ℓ represents the limit velocity of propagation of disturbances in the linear approximation. The direct

estimate of $\Lambda(v)$ may be affected by two technical problems: (i) a site-dependent normalization of the amplitude $|w_t^i|$ to cope with the different growth rates along the different lines $i = vt$; (ii) the increasing number of sites that must be updated. These problems can lead to serious difficulties, whenever the convergence towards the asymptotic shape is slow. Deissler and Kaneko [6] applied the usual algorithm for Lyapunov exponents in a moving reference frame. This approach is not affected by the above problems and it works perfectly in open-flow systems. However, in closed systems (like those ones we are currently investigating) boundary conditions strongly affect the evolution of Lyapunov vectors.

This further problem can be overcome by following the approach recently developed in [7]. The authors have introduced the specific Lyapunov exponent $\lambda(\mu)$, defined as the maximum growth rate of a perturbation w_t^i characterized by an exponential profile, $w_t^i = \exp(-\mu i) \cdot u_t^i$. The new variable u_t^i, assumed to satisfy periodic boundary conditions, obeys the evolution equation

$$u_{t+1}^i = f'(y_t^i)[\frac{\varepsilon}{2}\mathrm{e}^{-\mu} u_t^{i-1} + (1-\varepsilon)u_t^i + \frac{\varepsilon}{2}\mathrm{e}^{\mu} u_t^{i+1}] \quad . \tag{4}$$

The specific exponent $\lambda(\mu)$ can be estimated by applying the standard algorithm developed for the computation of Lyapunov spectra. The comoving exponent is related to $\lambda(\mu)$ through the Legendre trasform [7]

$$\Lambda(v) = \lambda(\mu) - \mu\lambda'(\mu) \quad ; \quad v = -\lambda'(\mu) \tag{5}$$

From the above equation,

$$v_\ell = -\lambda'(\overline{\mu}) \quad ,$$

where $\overline{\mu}$ is the μ-value such that the first of Eq. (5) vanishes. The derivative $\lambda'(\mu)$ can be estimated with a good accuracy by iterating a suitable recursive equation. After deriving Eq. (4) with respect to the parameter μ and introducing $z_t^i = (u_t^i)'$, we obtain

$$z_{t+1}^i = f'(y_t^i)[-\frac{\varepsilon}{2}\mathrm{e}^{-\mu} u_t^{i-1} + \frac{\varepsilon}{2}\mathrm{e}^{\mu} u_t^{i+1}] + f'(y_t^i)[\frac{\varepsilon}{2}\mathrm{e}^{-\mu} z_t^{i-1} + (1-\varepsilon) z_t^i + \frac{\varepsilon}{2}\mathrm{e}^{\mu} z_t^{i+1}] \tag{6}$$

The derivative of the specific exponent is then given by

$$\lambda'(\mu) = \frac{\sum_k z_{t+1}^k \cdot u_t^k}{(t+1)(\sum_k u_{t+1}^k \cdot u_{t+1}^k)^{1/2}} \tag{7}$$

The results of simulations performed for several η values are reported in Fig. 1 (dashed line). The linear propagation develops for $\eta > \eta_1$, when the maximum Lyapunov exponent becomes larger than zero.

The evolution of finite disturbances and the related nonlinear mechanisms can be investigated only in terms of a direct approach. Let us consider two initial configurations $\{x_t^i\}$, $\{y_t^i\}$ differing by order $\mathcal{O}(1)$ in a finite interval and coinciding outside. The corresponding configurations, generated after T time steps, are then compared to determine the left and right borders i_l, resp. i_r of the region where they differ more than a preassigned threshold θ. The propagation velocity $v_{n\ell}$ of finite disturbances is then defined as

$$v_{n\ell} = \lim_{T\to\infty} \frac{(i_r - i_l)}{2T} \quad .$$

Numerical simulations reveal a sizeable dependence of the actual value of $v_{n\ell}$ on the computational accuracy. For instance, upon increasing the number of significant digits from 8 to 64, the value of $v_{n\ell}$ increases by about 2 % for $\eta = 0.01$. This inaccuracy prevents a quantitative study of the scaling behaviour of the difference $\Delta v = v_\ell - v_{n\ell}$ for $v_\ell \to v_{n\ell}$. Such a difficulty has been circumvented by extrapolating the numerically exact value of $v_{n\ell}$ from the results of increasingly accurate simulations [8]. As a matter of fact, the finite-precision estimates of the velocity converge as $1/N_b^2$, where N_b is the number of significant digits. The asymptotic values of $v_{n\ell}$ are reported in Fig. 1 (solid line). Moreover, upon varying the value of the threshold θ from 1.10^{-4} to 0.1, the resulting variation of $v_{n\ell}$ is practically negligible.

Any infinitesimal perturbation w_t^i is exponentially amplified within the light-cone defined by v_ℓ. Therefore, w_t^i eventually becomes larger than any finite threshold θ, so that the nonlinear velocity $v_{n\ell}$ cannot be smaller than v_ℓ. This inequality is indeed satisfied in the simulations reported in Fig. 1, where two different "phases" are identified: for $\eta \geq \eta_c = 0.013$, $v_\ell = v_{n\ell}$ (I); for $\eta < \eta_c$, $v_{n\ell} > v_\ell$ (II). The former phase corresponds to the usual regime found in lattices of logistic and cubic maps [9]. The latter phase corresponds to a new regime, where the instability due to a positive Lyapunov exponent is not sufficient to account for the propagation of information along the chain. The

examples analysed in Refs. [1,2] represent a limit case of this scenario, where there is no linear instability at all.

The transition between the two regimes can be investigated from the dependence of the order parameter $\Delta v = v_\ell - v_{n\ell}$ versus $\eta_c - \eta$. The numerical results reported in Fig. 2, reveal a scaling behaviour of the type

$$\Delta v = (\eta_c - \eta)^\gamma \quad . \tag{8}$$

The optimal scaling behaviour is obtained for the above reported value of $\eta_c = 0.013$ yielding $\gamma \simeq 2$. Having we failed to derive even a heuristic explanation of this exponent, we have tried to reach a more complete understanding of this phemomenon by studying the fluctuations of the effective maximal comoving Lyapunov exponent $\tilde\Lambda$. The effective exponent $\tilde\Lambda$ corresponding to a velocity v over a time t is defined as

$$\tilde\Lambda \equiv \frac{1}{t} \log \left| \frac{y_t^i - x_t^i}{\delta} \right| \quad , \tag{9}$$

where $i = vt$ and $\{x_t^i\}$, $\{y_t^i\}$ represent two configurations which, at time $t = 0$, differ by $\delta_0^0 = \delta$ in the site $i = 0$, while coincide elsewhere. Moreover, let $P(\tilde\Lambda, v, t)$ denote the probability density to find $\tilde\Lambda$ in the interval $(\tilde\Lambda, \tilde\Lambda + d\tilde\Lambda)$. In the limit $\delta \to 0$ and $t \to \infty$, the whole information about the fluctuations of the Lyapunov exponent is conveyed by the multifractal spectrum

$$S(\tilde\Lambda, v) = \frac{\log(P(\tilde\Lambda, v, t))}{t} \quad . \tag{10}$$

The Lyapunov analysis has been performed for $\eta = 2 \cdot 10^{-3}$. For this parameter value, linear and nonlinear velocities are $v_\ell = 0.4184$, $v_{n\ell} = 0.5805$, respectively. We have computed the spectrum S for different velocities and time lags. The curves reported in Fig. 3 have been all obtained for $t = 40$; labels 1, 2 and 3 refer to $\delta = 10^{-7}$, 10^{-4} and 10^{-2}, respectively. Finally, the spectra reported in Fig. 3a correspond to $v = 0.7$, while those ones in Fig. 3b to $v = 0.5$.

If the amplitude of the perturbation is initially smaller than $\delta_c = \exp[-\tilde\Lambda_{max}(v)t]$ (where $\tilde\Lambda_{max}(v)$ is the maximum value assumed by the effective Lyapunov exponent $\tilde\Lambda$ at velocity v), then it will always obey a linear equation

during its evolution over the first (arbitrary) t time steps. In this case, one is simply estimating the multifractal spectrum of the comoving Lyapunov exponent Λ. On very general grounds, a bell shaped spectrum is expected. This is in fact what observed for the curves labelled with "1" in Fig. 3. If δ is increased above δ_c, some perturbations become of order $\mathcal{O}(1)$ and are then controlled by the full nonlinear equations. One could simply conjecture that all $\tilde{\Lambda}$ values greater than $\tilde{\Lambda}_u = -\log \delta / t$ are no longer detectable, while the rest of the spectrum remains unchanged. This is precisely what happens when the velocity v is larger than $v_{n\ell}$ as, for instance, in Fig. 3a. If instead, $v < v_{n\ell}$, a second peak appears at high Λ-values, when δ is increased (see Fig. 3b). For δ sufficiently "large", the new peak becomes higher than the old one, meaning that the nonlinear mechanism prevails onto the linear one.

The order of magnitude of the statistical error is much less than the amplitude of the oscillations observed at small $\tilde{\Lambda}$s: they are intrinsic fluctuations which are seen to decay slowly for increasing t. However, this finite-size phenomenon does not prevent a clear-cut distinction between the qualitative behaviour of the spectra observed in Fig. 3a and 3b. Therefore, we can conclude that the exchange of the two limits $\delta \to 0$ and $t \to \infty$ plays a crucial role in the phenomenon studied in this Letter. The usual definition of Lyapunov exponent is recovered if the former limit is taken first, while the nonlinear propagation of information is revealed only by taking first the latter limit.

In this Letter we have discussed two different mechanisms for information propagation. We have identified a regime where the prevailing mechanisms depends on the amplitude of the propagating perturbation. Whenever nonlinear processes eventually overtake the linear amplification, we expect different statistical properties of the associated spatio-temporal pattern. Finally, notice that the relevance of nonlinear mechanisms has been pointed out also in a standard CML of logistic maps, with reference to the predictability problem [10].

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Figure Captions

Fig. 1 Linear (dashed) and nonlinear (solid) velocities versus η. For the CML model (2) with $p_1 = 2.7$,$p_2 = 0.1$, $q_1 = 0$, $q_2 = 0.07$ and diffusive coupling $\varepsilon = 2/3$. The two arrows indicate the position of the parameters $\eta_1 = 4 \cdot 10^{-4}$ and $\eta_c = 0.013$.

Fig. 2 $\Delta v = v_\ell - v_{n\ell}$ versus $\eta_c - \eta$ ($\eta_c = 0.013$) reported in a log - log scale. The circles represent the actual point where the difference Δv have been calculated.

Fig. 3 Multifractal spectra $S(\tilde{\Lambda}, v)$ for $\eta = 2 \cdot 10^{-3}$ and $t = 40$: (a) $v = 0.7$; (b) $v = 0.5$. The labels 1, 2 and 3 correspond to $\delta = 10^{-7}$, 10^{-4} and 10^{-2}, respectively. The spectra have been obtained as an average over 200,000 different realizations.